Presenting frames - Part 1

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Abstract

This talk is a 3-part lecture series in which we present frames as distributive lattices satisfying the so-called infinite distibutive law. On one hand frames are viewed as Heyting algebras, on the other as generalized lattices of "opens". The latter view enables one to revisit many classical results of general topology - an exercise dubbed as "doing topology without points", "pointfree topology" or "pointless topology" - with the benefit, sometimes, of not having to rely heavily on choice principles.

Key words: complete lattice, frame, locale, sober space, spatial locale, sublocale.

We draw notions from topology, lattice theory and category theory.

Part 1

Lattices

Partially ordered set

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- If $a \leq b$ and $b \leq a$, then $a = b$. (Anti-symmetry)
- The pair $(A, \leq)$ called a partially ordered set (or simply, *poset*).

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- The set $\mathbb{Q}$ of rational numbers (fractions, including whole numbers $+$ve and -ve) is a poset.
- Given any set X, the power set $\mathcal{P}(X)$, with subset inclusion relation $\subseteq$, is a poset.

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- Similarly, $1 \in A$ is called the *top* element, if $a \leq 1$ for all $a \in A$.
- A given poset need not have the top nor bottom element: e.g. $\mathbb{N}$ has bottom element 0, but no top element; $\mathbb{Q}$ has neither top nor bottom elements.

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- Similarly, $\beta \in A$ called the supremum (least upper bound, or simply, *sup*) of S if β is an upper bound and whenever $b \in A$ is an upper bound, then $\beta \leq b$.

Lattice

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- If a lattice A has a top 1 and bottom element 0, then for the empty set $\emptyset \subseteq A$ we have $\sup(\emptyset) = 0$ and $\inf(\emptyset) = 1$.

Equational presentation

In a lattice A with bottom element 0 and top element 1, write $a \wedge b = \inf\{a, b\}$ and $a \vee b = \sup\{a, b\}$. Then the following properties hold for all $a, b, c \in A$:

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- $a \wedge b = b \wedge a$; $a \vee b = b \vee a$.
- $a \wedge a = a$; $a \vee a = a$.
- $a \wedge 1 = a$; $a \vee 0 = a$.

Lattice structure

One then realizes that a lattice A is in fact a five-tuple $(A, \wedge, \vee, 0, 1)$, where $\wedge$ and $\vee$ are binary operations satisfying the above equations.

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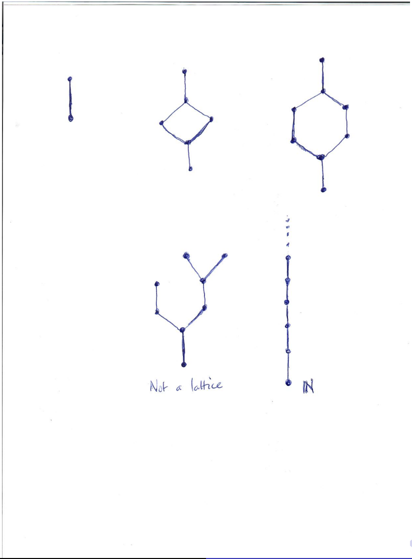
- We will frequently refer to $\wedge$ as *meet* operation and $\vee$ as *join* operation.

Examples



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- A is a *distributive lattice*, if for all $a, b, c \in A$,

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- In that case the following "dual" law is also satisfied:

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

Complementation

<u>Proposition</u>: In a distributive lattice A for any $a, b, c \in A$, there is at most one $x \in A$ such that

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- In case A is a distributive lattice, then, by the above proposition, a' is unique, and we write $a' = \neg a$.
- A distributive lattice A in which $\neg a$ exists for each $a \in A$ is called a *Boolean algebra*.

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- In a Boolean algebra A, the following de Morgan's laws hold:
  (i) $\neg(a \wedge b) = \neg\, a \vee \neg\, b$
  (ii) $\neg(a \vee b) = \neg\, a \wedge \neg\, b$.

Heyting algebra

A lattice A is called a *Heyting algebra* if for any $a, b \in A$, there is an element $(a \to b) \in A$ with the property that: for any $c \in A$,

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- If A is a Boolean algebra, then $(a \to b) = \neg\, a \vee b$.

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- The binary operation $\rightarrow$ is also known as the *Heyting implication*.

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- $a \to (b \vee c) \geq (a \to b) \vee (a \to c)$.

Complete lattice

A lattice A is said to be *complete* if for any subset $S \subseteq A$ we have $\bigvee S = \sup(S)$ exists as a member of A.
This is equivalent to the property that for any $S \subseteq A$, $\bigwedge S = \inf(S) \in A$.

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- A complete lattice A is said to satisfy the infinite distributive law if forny $a \in A$ and any $S \subseteq A$,

$$a \wedge (\bigvee S) = \bigvee \{a \wedge s \mid s \in S\}.$$